Lyapunov exponents and anomalous diffusion of a Lorentz gas with infinite horizon using approximate zeta functions

Per Dahlqvist

Mechanics Department

Royal Institute of Technology, S-100 44 Stockholm, Sweden

Abstract - We compute the Lyapunov exponent, generalized Lyapunov exponents and the diffusion constant for a Lorentz gas on a square lattice, thus having infinite horizon. Approximate zeta functions, written in terms of probabilities rather than periodic orbits, are used in order to avoid the convergence problems of cycle expansions. The emphasis is on the relation between the analytic structure of the zeta function, where a branch cut plays an important role, and the asymptotic dynamics of the system. We find a diverging diffusion constant $D(t) \sim \log t$ and a phase transition for the generalized Lyapunov exponents.

1 Introduction

Maybe the most well known measure of a chaotic system is the Lyapunov exponent. In the theory of chaotic dynamics one is of course interested in calculating this and similar quantities, either by finding analytical estimates or by devising effective calculation schemes but often one find oneself compelled to numerical simulation. This is unsatisfactory since it is not an easy task to extract information on the asymptotic behaviour from numerical data.

Various averages of chaotic systems are obtainable via transfer operators and and their Fredholm determinants or *zeta functions*. This is a beautiful formalism but the best results are obtained for a very restricted class of chaotic systems, namely those fulfilling Axiom-A. This is because Axiom-A guarantees nice analytical features of the zeta functions [1, 2, 3, 4] which enables fast convergent cycle expansions to deduce its leading zeros [5]. Applications of cycle expansions to non Axiom-A systems are not very successful [6, 7].

In this paper we will study a system which is far from the textbook 1-d Axiom-A map, namely the two-dimensional Lorentz gas on a square lattice. This is a Hamiltonian system with two degrees of freedom, continous time and with an infinite symbolic dynamics. We will demonstrate that zeta functions may be of great use even here. The key point is that we will avoid writing the zeta function in terms of periodic orbits as that would lead us to divergence problems that we could not handle. The price we will pay is that our zeta functions are no longer exact, but they are approximate in a sense that won't effect the leading zero very much. The averages we will compute are directly related to the motion of this leading zero with respect to variations of a parameter. However, matters will be complicated if there are non-analyticities in the vicinity of the leading zero, like branch cuts. This is, we believe, a generic feature of non Axiom-A systems. Such singularities will cause problems for cycle expansions but do carry important information

about the asymptotic dynamics. In the Lorentz gas there will be a branch cut connected to the existence of the infinite horizon. It will not prevent the Lyapunov exponent from being well defined but will yield a diverging diffusion constant. It will also imply a phase transition for the generalized Lyapunov exponents.

In section 2 we review the necessary theory. In section 3 we perform all calculations and we then end with some comments in section 4.

2 Theory

2.1 Lyapunov exponents and zeta functions

The largest Lyapunov exponent is defined by

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log |\Lambda(x_0, t)| \quad , \tag{1}$$

provided the limit exists and is independent of the initial point x_0 (except for a set of measure zero: namely the periodic points). $\Lambda(x_0, t)$ is the largest eigenvalue of the Jacobian along the trajectory starting at x_0 and evolving during time t. Using ergodicity this may be rewritten as a phase space average

$$\lambda = \frac{1}{t} \int \mu(dx) \log |\Lambda(x_0, t)| \equiv \frac{1}{t} < \log |\Lambda(x_0, t)| > \quad , \tag{2}$$

where $\mu(dx)$ is the invariant density.

We will now formulate the Lyapunov exponent in terms of evolution operators and zeta functions. Consider the following evolution operator acting on a phase space density $\Phi(x)$ according to

$$\mathcal{L}_w^t \Phi(x) = \int w(x, t) \delta(x - f^t(y)) \Phi(y) dy \quad . \tag{3}$$

It is essential for the further development that the weight $w(x, t)$ is multiplicative along the flow. We intend to evaluate the trace of this operator as that gives the expectation values of the weight (see appendix)

$$< w > = \lim_{t \to \infty} tr \mathcal{L}_w^t \quad . \tag{4}$$

The trace naturally turns out to be a sum over the periodic orbits

$$tr \mathcal{L}_w^t = \int w(x, t) \delta(x - f^t(x)) dx = \sum_p T_p \sum_{n=1}^{\infty} w_p^n \frac{\delta(t - nT_p)}{|det(1 - M_p^n)|} \quad , \tag{5}$$

where n is the number of repetitions of primitive orbit p, having period T_p, and M_p is the Jacobian (transverse to the flow). w_p is the weight associated with cycle p.

Zeta functions are introduced by observing that the trace may we written as

$$tr \mathcal{L}_w^t = \frac{1}{2\pi i} \int_{-\infty - ia}^{\infty - ia} e^{ikt} \frac{Z_w'(k)}{Z_w(k)} dk \quad . \tag{6}$$

For a Hamiltonian system with two degrees of freedom the zeta function reads [8]

$$Z_w(k) = \prod_p \prod_{m=0}^{\infty} \left(1 - w_p \frac{e^{-ikT_p}}{|\Lambda_p| \Lambda_p^m} \right)^{m+1} \quad , \tag{7}$$

where Λ_p is the expanding eigenvalue of M_p.

Such infinite products over periodic orbits are often troublesome because they diverge beyond the first (nontrivial) zero. It is essential that the constant a in eq. (6) is sufficiently large so that the products converge. In our subsequent calculation it suffices if a is small and positive.

Cycle expansions of the zeta functions generally have better convergence properties since they converge up to the first singularity [5]. The zeta function is entire for Axiom-A system which makes cycle expansions very successful for this special case. In this paper we will consider cases where the leading zero is also a singularity (branch point) so a cycle expansion will not converge even there and the zeta function is rather useless as it stands. We return to these problems in section 2.3.

We must now find a weight w appropriate for computing the Lyapunov exponent. The quantity whose average we are going to study is $log|\Lambda(x_0, t)|$ which is certainly not multiplicative. In one dimension we can study the average of the multiplicative weight $w = |\Lambda(x_0, t)|^\tau$ and obtain the Lyapunov exponent by differentiation

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \frac{d \, tr\mathcal{L}_\tau^t}{d\tau}|_{\tau=0} \quad . \tag{8}$$

The problem to find the appropriate weighted operator in more dimensions is nontrivial and the problem was recently solved [9]. But for our purposes it will suffice to insert the weight w above into the operator $\mathcal{L}_w^t$ just defined. The modification in [9] to make w exactly multiplicative is complicated but minor and we don¡t expect it to affect the leading zero of the zeta function, and its vicinity, in which we are interested.

The leading zero $k_0(\tau) = -i \cdot h(\tau)$ is always on the negative imaginary axis, if $\tau > 0$. It provides the leading asymptotic behaviour of the trace provided that there is a gap until next zero or singularity. Then we have

$$\lambda = \frac{d}{d\tau} h(\tau) \quad . \tag{9}$$

Generalized Lyapunov exponents [10] $\lambda(\tau)$ are defined by considering the scaling behaviour of $< |\Lambda(x_0, t)|^\tau >$:

$$< |\Lambda(x_0, t)|^\tau > = \lim_{t \to \infty} tr\mathcal{L}_\tau^t = e^{\lambda(\tau)\tau t} \quad , \tag{10}$$

so that

$$\lambda(\tau) = \frac{h(\tau)}{\tau} \quad , \tag{11}$$

provided again that the leading zero is isolated. The ordinary Lyapunov exponent is recognized as the limit $\lambda = \lim_{\tau \to 0} \lambda(\tau)$.

2.2 Diffusion coefficients and zeta functions

We will consider the Lorentz gas obtained by unfolding the Sinai billiard. The coordinate in the unfolded system is called $\hat{x}$. The corresponding vector in the billiard (or the *unit cell*) is x. They are related by translation $\hat{x} - x \in T$ where T is the group of translations building up the Lorentz gas from the unit cell.

The diffusive properties can be extracted from the average

$$< e^{\beta \cdot (\hat{f}^t(x_0) - x_0)} >_{x_0} \quad . \tag{12}$$

The average is taken over one unit cell. Again we must perform the trick to introduce a multiplicative weight and then by differentiation extract the average in which we are interested

It was demonstrated in ref [11] that this average can be computed by considering the dynamics in the unit cell only. This is obtained by inserting the weight

$$w(x, t) = e^{\beta \cdot (\hat{f}^t(x) - x)} \tag{13}$$

into the evolution operator (3). The diffusion constant is now given by

$$D = \lim_{t \to \infty} \frac{1}{\nu t} \sum_{i=1}^{\nu} < (\hat{f}(x_0) - x_0)^2 >_{x_0} = \lim_{t \to \infty} \frac{1}{\nu t} \sum_{i=1}^{\nu} \frac{\partial^2}{\partial \beta_i^2} tr\mathcal{L}_\beta^t \quad . \tag{14}$$

where the sum is taken over spatial components of the 2ν dimensional phase space.

2.3 Approximate zeta functions

In some recent publications we have investigated a way of approximating zeta functions for intermittent systems [7, 12, 13]. We call it the BER-approximation after the authors of ref [14]. In an intermittent system laminar intervals are interrupted by chaotic outbursts. Let Δ_i be the time elapsed between two successive entries into the laminar phase. The index i labels the i'th interval. Provided the chaotic phase is *chaotic enough*, the lengths of the intervals Δ_i are presumed uncorrelated, and Δ may be considered as a stochastic variable with probability distribution $p(\Delta)$. The zeta functions (unit weight $w = 1$) may then be expressed in terms of the Fourier transform of $p(\Delta)$

$$Z(k) \approx \hat{Z}(k) \equiv 1 - \int_0^\infty e^{-ik\Delta} p(\Delta) d\Delta \quad . \tag{15}$$

Due to the normalization of $p(\Delta)$ we see that leading zero $k_0 = 0$ is by construction exact because of probability conservation.

In order to compute the probability distribution we introduce a surface of section (SOS). This should, according to the BER prescription be put on the border between the laminar and chaotic phase. We call the phase space of the SOS Ω and its coordinates x_s. The flight time to the next intersection is then a function of x_s: $\Delta_s(x_s)$. The probability distribution then reads

$$p(\Delta) = \int_\Omega \delta(\Delta - \Delta_s(x_s)) \mu(dx_s) \quad , \tag{16}$$

where $\mu(dx_s)$ is the invariant density, which is uniform $\mu(dx_s) = dx_s / \int_\Omega dx_s$ for Hamiltonian ergodic system, assuming of course that the SOS coordinates are canonically conjugate.

It is straightforward to include weights in this formalism, like e.g. $w = |\Lambda(x_0)|^\tau$. The local expansion factor $\Lambda_s(x_s)$ is also a function of x_s. The zeta function is then related to a generalized distribution

$$p_\tau(\Delta) = \int_\Omega |\Lambda_s(x_s)|^\tau \delta(\Delta - \Delta_s(x_s)) dx_s \quad . \tag{17}$$

The zeta function $\hat{Z}_\tau(k)$ is obtained by inserting $p_\tau(\Delta)$ into (15).

3 Application to the Lorentz gas

We now begin our study of the Lorentz gas on a square lattice. The lattice spacing is unity, the disk circular with radius $R < 1/2$, and the point particle bouncing around in this array has unit velocity. It is the unit cell of this system, the Sinai billiard [15] whose dynamics we will study and this is indeed an intermittent system; there exist periodic orbits with arbitrary small Lyapunov exponents $\log \Lambda_p / T_p$. The disk will define our SOS. We use the two angles ϕ and α defined in fig 1a as coordinates. The normalized measure is then $dx_s = d\phi \, d(\sin \alpha)/4\pi$. Consider now a segment of the trajectory between two disk collisions. This segment can be labeled according to the disk that would be hit in the unfolded system, the label $\mathbf{q} = (n_x, n_y)$ is the associated lattice vector [12]. It is easy to realize that only disks associated with coprime lattice vector may occur.

The purpose is now to apply the BER approximation to this system and compute Lyapunov exponents and diffusion constants, cf refs [12, 13].

3.1 Calculation of $p_\tau(\Delta)$

In this section we will derive the following approximate expression for $p_\tau(\Delta)$

$$p_\tau(\Delta) \approx \begin{cases} \frac{4}{3} \frac{\Gamma(\frac{2-\tau}{2})^2}{\Gamma(2-\tau)} R^{1-\tau} \Delta^\tau & \Delta \leq 1/2R \\ \frac{4}{3} \frac{\Gamma(\frac{2-\tau}{2})^2}{\Gamma(2-\tau)} \frac{2^{\tau/2-3} R^{-\tau/2-2}}{\Delta^{3-3\tau/2}} & \Delta > 1/2R \quad . \end{cases} \tag{18}$$

Figure 1: a) The Sinai billiard with definitions of the variables ϕ and α. b) The unfolded system with free directions (corridors) indicated. c) The region of integration.

This expression was used already in ref [13] but as it plays a central role in this paper we present its derivation in some detail. Complementary details may be found in refs [12, 13].

We start from expression (17) for $p_\tau(\Delta)$. First we partition the SOS into subsets $\Omega_{\mathbf{q}}$ where $\Omega_{\mathbf{q}}$ is the part of Ω for which the trajectory hits disk $\mathbf{q}$. Then we smear the distribution, that is, we replace the delta function with some extended distribution δ_σ. The exact form of this function is irrelevant, the only thing we assume is that the width is big: $\sigma \gg 1$. We will be interested in the behaviour of the leading zero of $\hat{Z}_\tau(k)$ for small τ. This zero is then close to the origin and it is evident that smearing of $p_\tau(\Delta)$ will only have minor effect on this zero.

We now have

$$p_\tau(\Delta) = \sum_{\mathbf{q}} \int_{\Omega_{\mathbf{q}}} |\Lambda_s(x_s)|^\tau \delta_\sigma(\Delta - \Delta_s(x_s)) dx_s \quad . \tag{19}$$

The large width σ allow us to move the smeared delta function to the left of the integral sign because the variation of $\Delta_s(x_s)$ over $\Omega_{\mathbf{q}}$ is of the order $\sim R$ and we have $\sigma \gg 1 > R$.

$$p_\tau(\Delta) = \sum_{\mathbf{q}} \delta_\sigma(\Delta - q) \int_{\Omega_{\mathbf{q}}} |\Lambda_s(x_s)|^\tau dx_s \equiv \sum_{\mathbf{q}} \delta_\sigma(\Delta - q) a_{\mathbf{q}}(\tau) \quad . \tag{20}$$

We have chosen the length of the lattice vector $|\mathbf{q}| \equiv q$ as a mean value of $\Delta_s(x_s)$. The local expansion factor is $|\Lambda_s(x_s)| = 2\Delta_s(x_s)/R\cos(\alpha)$.

It is easily shown that the phase space area taken up by disk $\mathbf{q}$ is given by the inequality

$$| \frac{q}{R} \sin(\phi - \theta_{\mathbf{q}} - \alpha) + \sin(\alpha) | < 1 \quad , \tag{21}$$

where $\theta_{\mathbf{q}}$ is the polar angle of the lattice vector $\mathbf{q}$. Generally parts of this region are eclipsed by disks closer to the origin. So in order to find $\Omega_{\mathbf{q}}$ one has to subtract these. We will focus on the limit of small R which gives the more easily handled inequality

$$|\frac{q}{R}(\phi - \theta_{\mathbf{q}} - \alpha) + \sin(\alpha)| < 1 \quad . \tag{22}$$

Let us begin with the limit of small Δ. In ref [12] it is shown that if disk $\mathbf{q}$ lies within a certain radius: $q < 1/2R$, there are no eclipsing disks in front of it, and expression (22) may be used directly, and the integral is easily calculated:

$$a_{\mathbf{q}}(\tau) = (\frac{2q}{R})^\tau \frac{1}{4\pi} \int_{\Omega_{\mathbf{q}}} \cos^{\tau+1}(\alpha) d\alpha \, d\phi \approx \frac{1}{4\pi} \frac{\Gamma(\frac{2-\tau}{2})^2}{\Gamma(2-\tau)} (\frac{R}{q})^{1-\tau} \quad . \tag{23}$$

In order to find an approximate expression for $p_\tau(\Delta)$ we must know the density of coprime lattice points, i.e. the average number $d_c(r)dr$ of such points having a distance between r and $r + dr$ from the origin. In ref [13] this was, to leading order, found to be $d_c(r) \approx \frac{16\pi}{13} r$. This yields

$$p_\tau(\Delta) = \sum_{\mathbf{q}} \delta_\sigma(\Delta - q) a_{\mathbf{q}}(\tau) = \frac{16}{13} \frac{\Gamma(\frac{2-\tau}{2})^2}{\Gamma(2-\tau)} R^{1-\tau} \Delta^\tau \quad . \tag{24}$$

Obviously the disk radius has to be small for this to apply.

Next we consider the opposite limit $\Delta \gg 1/2R$. For each (coprime) disk $\mathbf{q}$ fulfilling $q < 1/2R$ there are two (or one, depending on symmetry) transparent corridor in the direction $\mathbf{q}$ [12, 16], see fig 1b. Far beyond this critical radius the accessible disks will be those adjacent to the corridors. (They still have to coprime so they will lie on one side of the corridor only, see fig 1b). We will discover that this will lead to a power law decay of $p(\Delta)$. We will be interested in the particular power (as a function of τ) and not the prefactor. For that reason we perform our calculation in the corridor having direction vector $(1,0)$ as all corridors provide the same power. In this corridor the accessible disks are the ones being labeled $(n,1)$. Disk $(n,1)$ is shadowed by $(1,0)$ and $(n-1,1)$. We need to evaluate the integral (cf eq (23))

$$j_n \equiv \int_{\Omega_{\mathbf{q}=(n,1)}} \cos^{\tau+1}(\alpha) d\alpha \, d\phi \quad . \tag{25}$$
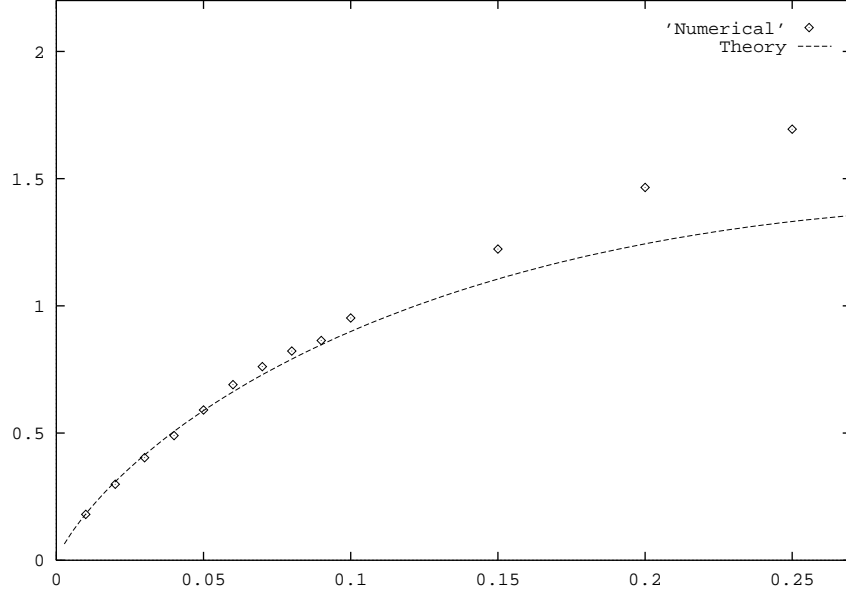
Figure 2: Lyapunov exponent versus disk radius according to numerical simulation and eq (32).

It is more convenient to consider the sum

$$J_n \equiv \sum_{i=n}^{\infty} j_i = \int_{\bigcup_{i=n}^{\infty} \Omega_{\mathbf{q}=(n,1)}} \cos^{\tau+1}(\alpha) d\alpha \ d\phi \ , \tag{26}$$

because this integral has support from the triangular region in fig 1c (it is a triangle only in the limit $n \to \infty$ of course).

From eq (22) one can deduce that the base length (in the ϕ direction) of this triangle scales as $\sim 1/n$ and the height (in the α direction) as $1/\sqrt{(n)}$. A short calculation now yields that $J_n \sim 1/n^{2-\tau/2}$. Differentiation gives $j_n \sim 1/n^{3-\tau/2}$. The fact that $l_{\mathbf{q}} \sim n$ together with eq. (23) implies that the $a_{\mathbf{q}}$'s decay as $\sim 1/n^{3-3\tau/2}$. The density of accessible disks is uniform in Δ (since they lie along the corridors) so our final result is

$$p(\Delta) \sim \frac{1}{\Delta^{3-3\tau/2}} \quad \Delta \gg \frac{1}{2R} \ . \tag{27}$$

This power law does not depend on the small R limit.

The prefactor of eq (27) is determined by demanding that $p_\tau(\Delta)$ is continous at $\Delta = 1/2R$. In order to get correct normalization for $\tau = 0$ we multiply the entire $p_\tau(\Delta)$ by 13/12. This approximation means a rather crude neglect of the of the transitional behaviour at $\Delta \approx 1/2R$. In the following we must be very cautious when using it since some results may be more sensible to our approximation than other.

3.2 Lyapunov exponents

The approximate zeta function we are going to work with is obtained from (18)

$$\begin{aligned} \hat{Z}_\tau(k) \approx \ & 1 - \int_0^{\infty} e^{-ik\Delta} p_\tau(\Delta) = \\ & 1 - \frac{4}{3} \frac{\Gamma(\frac{2-\tau}{2})^2}{\Gamma(2-\tau)} \frac{R^{-2\tau}}{2^{1+\tau}} (z^{-1-\tau} \gamma(\tau+1,z) + z^{2-3\tau/2} \Gamma(2+3\tau/2,z)) \ , \end{aligned} \tag{28}$$

where $z = \frac{ik}{2R}$. The functions $\gamma(a, z)$ and $\Gamma(a, z)$ are incomplete gamma functions [17]. Expanding this to first order in τ gives

$$
Z = \quad \left\{ z + \frac{z^2}{3}(\log z + \gamma - \frac{11}{6}) \ldots \right\} + \\
\left\{ (-\frac{7}{12} + \log(2R^2)) + (\frac{11}{6} + \log(2R^2))z \ldots \right\} \tau \quad \ldots
\tag{29}
$$

The derivation of this expansion is rather lengthy but the only step that is slightly tricky is the expansion of the incomplete gamma function $\Gamma(a, z)$ near an integral power a. We have for convenience thrown away all indices of the zeta function and use strict equality sign but we must not forget that we work with an approximation of an approximation of the exact zeta function.

We must now proceed with some care since the leading zero is not isolated, a branch cut along the negative real z-axis reaches all the way up to it (we have choosen the principal branch of the logarithm).

We are interested in the following derivative of the trace, cf eq (8)

$$
\frac{d}{d\tau} \, tr\mathcal{L}^t = \frac{d}{d\tau} \frac{1}{2\pi i} \int e^{zt'} \frac{d}{dz} \log Z \, dz \mid_{\tau=0} = \frac{1}{2\pi i} \int e^{zt'} \frac{d}{d\tau} \frac{d}{dz} \log Z \mid_{\tau=0} \, dz \quad,
\tag{30}
$$

where we have differentiated inside the integral sign. We have now formulated eq (6) in the rescaled and rotated $z - plane$ using rescaled time $t' = 2Rt$. The function to be Fourier transformed is

$$
\frac{d}{d\tau} \frac{d}{dz} \log Z \mid_{\tau=0} = (\frac{7}{12} - \log(2R^2)) \frac{1 + \frac{z}{3}(2 \log z + 2\gamma - \frac{5}{6}) \ldots}{z + \frac{z^2}{3}(\log z + \gamma - \frac{11}{6})^2 \ldots} = (\frac{7}{12} - \log(2R^2))(\frac{1}{z^2} + \frac{1}{3z} \ldots) \quad,
\tag{31}
$$

where we have kept only those terms yielding the leading term and the first correction. Collecting it all together yields

$$
\lambda = \lim_{t \to 0} 2R(\frac{7}{12} - \log(2R^2))(1 + \frac{1}{6Rt} \ldots) = 2R(\frac{7}{12} - \log(2R^2)) \quad.
\tag{32}
$$

The limiting value is due to the behaviour of the zero but the power law correction are due to the fact that it sits on a branch point. The particular size of the first correction is not very accurate, as we will realize after reading section 3.4 and 3.5, but the important thing to bear in mind is that there exist slowly decaying corrections indicating slow convergence of the Lyapunov exponent in numerical computations.

In fig 2 we compare numerical results on the Lyapunov exponent with our expression $\lambda = 2R(\frac{7}{12} - \log(2R^2))$. The numerical values, from [18], are calculated for rather large disk radii, where we should not expect much agreement a priori. Nevertheless our estimate is only 5% wrong when $R = 0.1$. The reason why the the numerical values exceed our estimates for large R is easily understood. This is because the disk faces (in the unfolded system) come closer to each other so that taking the length of the relevant lattice vector (as we did) overestimates the time of flight between them.

The Lyapunov exponent of the corresponding Poincaré map with the disk defining the SOS is related to the Lyapunov exponent of the flow according to [19] $\lambda_{map} = \lambda < \Delta_s > \approx \lambda/2R \approx (\frac{7}{12} - \log(2R^2))$ where $< \Delta_s >$ is computed by means of expression (18). We see that $\lambda \to -2 \log(R) + 7/12 - \log 2$ when $R \to 0$ which agrees with the conjectured limit [20, 21]: $\lambda \to -2 \log(R) + C + O(R)$. Indeed we have analytically found an estimate of the constant $C \approx 7/12 - \log 2 \approx -0.110$. This is very close to the numerical value found by [20], as far as we can extract it from fig 2 in ref [20]).

In fig 3 we plot the generalized Lyapunov exponents for different disk radii. Note that when $\tau > 0$ the leading zero is indeed isolated and we do not have to worry about the cut. The position of the zero is computed numerically. $\lambda(1)$ is the topological entropy which tend to a finite limit when $R \to 0$ [12], whereas $\lambda(0) \to 0$ as $R \to 0$. When $\tau < 0$ the branch cut itself will provide the leading behaviour of the trace - a power law [12, 13], and the generalized Lyapunov exponent will be zero. This means that $\lambda(\tau)$ cannot be analytic at $\tau = 0$. This is referred to as a phase transition [10, 22].
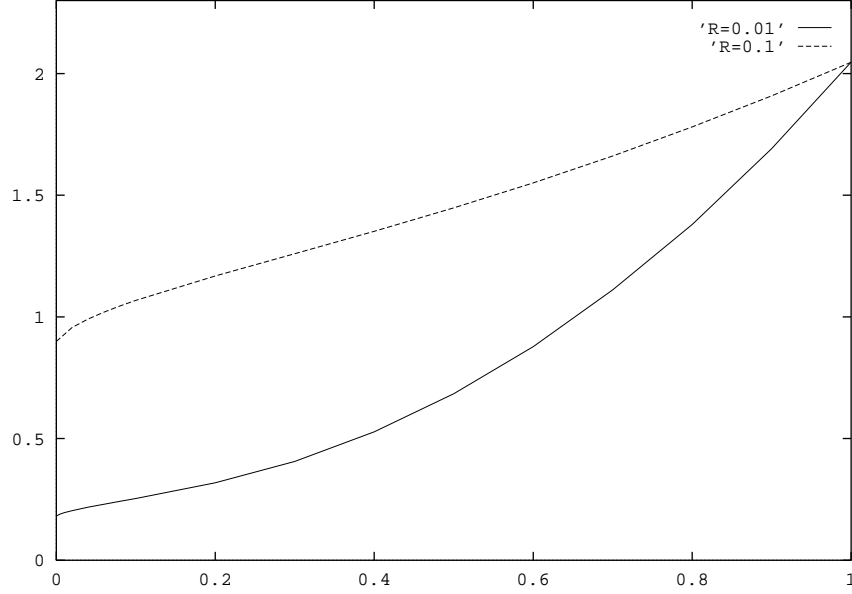
Figure 3: Generalized Lyapunov exponents $\lambda(\tau)$ versus τ for two disk radii.

3.3 A first calculation of the diffusion constant

In order to study diffusion we calculate the generalized probability distribution $p_w(\Delta)$ using appropriate weight (13). However, we only keep the spatial components of β yielding the two dimensional vector $\boldsymbol{\beta}$. As we study smeared $p_{\boldsymbol{\beta}}(\Delta)$ it suffices to approximate the spatial part of $(\hat{f}^t(x) - x)$ with the lattice vector $\mathbf{q}$. So, we must now compute the generalized probability distribution $p_{\boldsymbol{\beta}}(\Delta)$ a la section 3.1 but using the weight $\exp(\boldsymbol{\beta} \cdot \mathbf{q}) = \exp(\beta q cos(\phi_\beta - \phi_q)$ where we have written $\mathbf{q} = q(\cos(\phi_q), \sin(\phi_q))$ and $\boldsymbol{\beta} = \beta(\cos(\phi_\beta), \sin(\phi_\beta))$. We can use the result of section 3.1 to some extent, since we realize, after inspecting eq (17), that we can make the following factorization

$$p_{\boldsymbol{\beta}}(\Delta) = p_0(\Delta) \frac{\int e^{\Delta \beta cos(\phi_q - \phi_\beta)} d\phi_q}{\int d\phi_q} \tag{33}$$

To know the support of this integral we need to know about the angular distribution of accessable disks for a particular value of $q = \Delta$. To this end we introduce the following simplifying assumptions

1. the coprime lattice points are distributed isotropically.

2. if $q > 1/2R$ we assume only four corridors with ϕ_q equal to one of 0, $\pi/2$, π or $3\pi/2$.

Assumption 2 means a rather brute neglect of the transitional behaviour at $\Delta \approx 1/2R$. We neglect the important fact the number of corridors grows when $R \to 0$. We comment on this in section 3.5. The important thing now is that we preserve the symmetry in order to prevent net drift.

We first consider the case $q < 1/2R$. Using assumption 1 above gives

$$p_{\boldsymbol{\beta}}(\Delta) = \frac{4}{3} R \frac{1}{2\pi} \int_0^{2\pi} e^{\beta \Delta \cos \phi} d\phi = \frac{4}{3} R I_0(\beta \Delta) = \frac{4}{3} R(1 + \frac{\beta^2 \Delta^2}{4} \ldots) \quad \Delta < \frac{1}{2R} \quad , \tag{34}$$

where I_0 is a modified Bessel function In order to calculate the zeta function we need the Fourier transform of this

$$\int_0^{1/2R} p_{\boldsymbol{\beta}}(\Delta)e^{-ik\Delta}d\Delta = \frac{2}{3}(1 - z/2 + z^2/6 \ldots) + \frac{\beta'^2}{6}(1/3 - z/4 + z^2/10 \ldots) \quad , \ldots \tag{35}$$

where we have used rescaled variables $z = ik/2R$ and $\boldsymbol{\beta}' = \boldsymbol{\beta}/2R$.

Next we consider the limit $q > 1/2R$. We find, using the second assumption above

$$p_{\boldsymbol{\beta}}(\Delta) = \frac{2}{3(2R)^2}\frac{1}{\Delta^3}(e^{\beta'_x\Delta} + e^{-\beta'_x\Delta} + e^{\beta'_y\Delta} + e^{-\beta'_y\Delta}) \quad \Delta > \frac{1}{2R} \quad . \tag{36}$$

Fourier transforming this gives

$$\int_{1/2R}^{\infty} p_{\boldsymbol{\beta}}(\Delta)e^{-ik\Delta}d\Delta = \frac{1}{6}(E_3(z - \beta'_x) + E_3(z + \beta'_x) + E_3(z - \beta'_y) + E_3(z + \beta'_y)) \quad . \tag{37}$$

The resulting zeta function can now be expanded

$$\begin{aligned}
Z_{\boldsymbol{\beta}} = 1 - \int_0^{\infty} p_{\boldsymbol{\beta}}(\Delta)e^{-ik\Delta}d\Delta = \quad &\frac{1}{6}(\gamma - \frac{11}{6})\beta'^2 + z(1 - \frac{1}{8}\beta'^2) + z^2\frac{1}{3}(\gamma - \frac{11}{6}) + \\
&\frac{1}{12} \quad [(z - \beta'_x)^2\log(z - \beta'_x) + (z + \beta'_x)^2\log(z + \beta'_x) + \\
&(z - \beta'_y)^2\log(z - \beta'_y) + (z + \beta'_y)^2\log(z + \beta'_y)] \\
&\ldots
\end{aligned} \tag{38}$$

According to sec 2.2, we are interested in the following quantity

$$\begin{aligned}
(\frac{\partial^2}{\partial\beta_x^2} + \frac{\partial^2}{\partial\beta_y^2})tr\mathcal{L} \mid_{\beta'=0} = \quad &(\frac{\partial^2}{\partial\beta_x^2} + \frac{\partial^2}{\partial\beta_y^2})\frac{1}{2\pi i}\int e^{zt'}\frac{d}{dz}\log Z \, dz \mid_{\beta=0} \\
= \quad &\frac{1}{(2R)^2}\frac{1}{2\pi i}\int e^{zt'}(\frac{\partial^2}{\partial\beta_x'^2} + \frac{\partial^2}{\partial\beta_y'^2})\frac{d}{dz}\log Z \mid_{\beta=0} \, dz \quad .
\end{aligned} \tag{39}$$

We now proceed in the same way as in sec 3.2. We want to determine the asymptotic behaviour of the fourier transform of

$$(\frac{\partial^2}{\partial\beta_x'^2} + \frac{\partial^2}{\partial\beta_y'^2})\frac{d}{dz}\log Z \mid_{\boldsymbol{\beta}=0} \sim \frac{2}{3z^2}(\frac{4}{3} - \log z - \gamma) \quad . \tag{40}$$

To this end we now need the following integral

$$\frac{1}{2\pi i}\int \frac{\log z}{z^2}e^{zt'}dz = t'(1 - \gamma - log t') \quad . \tag{41}$$

In deriving this it is convenient to use contour integration and let the contour encircle the negative real z axis. We thus find the following diverging diffusion constant ($D = \lim_{t\to\infty} D(t)$)

$$D(t) = \frac{1}{2t}(\frac{\partial^2}{\partial\beta_x^2} + \frac{\partial^2}{\partial\beta_y^2})tr\mathcal{L}^t \mid_{\beta'=0} = \frac{1}{18R}(1 + 3log(2Rt)) \quad . \tag{42}$$

This logarithmic divergence of the diffusion constant agrees with ref [16] but the prefactor is not correct. The computation will be refined in the section 3.5. The important thing to learn from this section is that the Laplacian exposes the logarithm in the series expansion of the zeta function. So we now know where to focus our attention, namely at the tail of $p_{\boldsymbol{\beta}}(\Delta)$.

3.4   A closer look at the tail

At the end of section 3.1 we aimed at finding a good approximation for $p_\tau(\Delta)$ for the whole range $1/2R < \Delta < \infty$. In this section we will make a more careful investigation of the $\Delta \to \infty$ tail of $p_\tau(\Delta)$. We restrict ourselves to the case $\tau = 0$ and will compute the limit $\lim_{\Delta\to\infty} \Delta^3 p_0(\Delta)$ for small disk radii.

Let us look at the corridor with direction vector $\mathbf{q} = (n_x, n_y)$ where n_x and n_y are coprime. Suppose for a moment that $\mathbf{q}$ lies in the first octant so that n_x and n_y positive and $n_x \geq n_y$. The accessible disks in this corridor are the ones labeled $\mathbf{q}' + n\mathbf{q}$ and $\mathbf{q}'' + n\mathbf{q}$ where $\mathbf{q}'$ and $\mathbf{q}''$ are the predecessor and successor in the Farey sequence of order n_x [12]. A calculation analogous to the one at the end of section 3.1 now gives the following expression for $a_q(0)$ (as defined in eq (23))

$$a_{\mathbf{q}'+n\mathbf{q}} = \frac{1}{2\pi} \frac{2qR + \frac{1}{2qR} - 2}{q^2 n^3} + O(1/n^4) \tag{43}$$

and the same holds for the sequence $q'' + nq$. The contribution to $p_0(\Delta)$ is

$$\sum_n a_{q'+nq}\delta(\Delta - |q' + nq|) = \frac{1}{\Delta^3}\frac{1}{2\pi}(2qR + \frac{1}{2qR} - 2) \tag{44}$$

To compute the tail of $p_0(\Delta)$ we need to sum over all coprime $\mathbf{q}$ such that $q < 1/2R$. We restrict the summation to the first quadrant $\mathbf{q} \in S$

$$S = \{\mathbf{q} = (n_x, n_y)|n_y > 0;\ n_x \geq 0;\ (n_x, n_y) = 0;\ \sqrt{n_x^2 + n_y^2} < 1/2R\} \tag{45}$$

and multiply the result by four, to account for all four quadrants, and then by two, to account for both $\mathbf{q}'$ and $\mathbf{q}''$ defined above. The result is

$$p_0(\Delta) \sim \frac{4}{\pi\Delta^3} \sum_{\mathbf{q}\in S}(2qR + \frac{1}{2qR} - 2) + O(1/\Delta^4) \tag{46}$$

The nontrivial sum above is treated by Bleher [16] and we insert the small R limit of the sum and obtain

$$p_0(\Delta) \rightarrow \frac{1}{\Delta^3 2\pi^2 R^2} + O(1/\Delta^4) \tag{47}$$

Suppose that we expand the unweighted zeta function Z_0 into the more general series

$$Z_0 = \sum_{i=1}^{\infty} a_i z^i + \sum_{i=2}^{\infty} b_i z^i \log z \tag{48}$$

For the approximation of $p(\Delta)$ worked out in section 3.1 we have $b_i = 0$ for $i \geq 3$. The correct value of b_2 may be computed from eq (47) (using e.g. standard expansions of exponential integrals) and is found to be $b_2 = 1/\pi^2$ which differs considerably from the result in section 3.1. The reason for this error is the absence of a smooth transition at $\Delta = 1/2R$ in eq (18). The crossover behaviour is slow as indicated from the $O(1/\Delta^4)$ term above. Generally we expect the asymptotics of the tail to look like $p(\Delta) \sim \sum_{n\geq 3} c_n/\Delta^n$ so that there now may exist nonzero b_i of any order.

3.5   A second calculation of the diffusion constant

One could suspect that the coarse assumptions, especially assumption 2 in section 3.3, induce an error in the prefactor of super diffusion. We will now demonstrate that the prefactor is robust against refinements of the this assumption. A better estimate of $p_\beta(\Delta)$ is obtained by replacing $exp(\beta_x'\Delta) = exp(\beta'\Delta\cos(\phi_\beta))$ in eq (36) with the averaged quantity

$$\frac{1}{2c/\Delta} \int_{-c/\Delta}^{c/\Delta} e^{\beta'\Delta\cos(\phi_\beta - \phi_q)}d\phi_q \tag{49}$$

The integration range decreases like $1/\Delta$ since the width of the corridor is constant. For sufficiently large Δ we can take $\beta'\cos(\phi_\beta - \phi) \approx \beta_x' + \beta_y'\phi$ and find that the integral above is

$$e^{\beta_x'\Delta}(1 + O(\beta_y'^2)) \tag{50}$$

The correction is $O(\beta'^2)$ and has no effect on $D(t)$, as we anticipated.

We also reduced the number of corridors to four, irrespective of the disk radius. As long as we are not interested in the angular properties this is easily justified by studying how the laplacian acts on (38).

We are led to the conclusion that the prefactor of super diffusion depends solely on the coefficient b_2 of the unweighted zeta function (48) according to $D(t) = \frac{b_2}{2R} \log t$. Inserting the correct value of b_2 from the previous section we arrive at

$$D(t) = \frac{1}{2R\pi^2} \log t \tag{51}$$

which is indeed the exact result [16].

Again there are slowly decaying corrections to $D(t)$ and it is not surprising that the numerical detection of this diffusion law has eluded serious attempts [23].

4   Concluding remarks

The main advantage of the approximation outlined in this paper is its apparent simplicity. It is far more easy to apply than a proper cycle expansion would be, taking all the (infinite number of) pruning rules into account, and it gives a very good description of the leading zero. This zero and its vicinity, was our only interest in this paper and we could work out a very coarse description of $p(\Delta)$.

In ref. [12] we demonstrated that refinement of the expression for $p(\Delta)$ down to a scale $\sim R$ also accounted for a good description of the trace down to this scale. This means that some nonleading zeros may also be extracted within the BER approximation. However, in order to compute higher order spectra, like semiclassical spectra, one needs to take the correlation between laminar intervals into account. The way to do this would perhaps be to combine the knowledge of the asymptotics on the periodic orbits, as obtained from the present approximation, with a set of explicitly computed periodic orbits.

The major drawback of the BER approximation is the lack of error term; so far we have no bounds on the errors in our computations. It is of course highly desirable to work out such bounds which would put the present considerations on mathematically firm ground.

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Appendix

In this appendix we will verify equation (4) $< w >= \lim_{t \to \infty} tr \mathcal{L}_w^t$.

The invariant density may be expressed in terms of periodic orbits as $\mu(dx) = \lim_{t \to \infty} \mu_t(dx)$ where

$$\mu_t(dx) = dx \sum_{p \in p.p.o.} \sum_{n=1}^{\infty} \frac{\delta(t - nT_p)}{|det(1 - M_p^n)|} \int_{x' \in p} \delta(x - x') \frac{dx'}{v} \tag{52}$$

where v is the speed at x.

The invariant density is a eigenfunction of the evolution operator ($w = 1$) corresponding to the leading eigenvalue (equal to one). This requirement is satisfied by $\mu_t(dx)$ for any t. However, $\mu_t(dx)$ is only a smooth function in the limit $t \to \infty$ (we refrain from showing this). It follows from the boundedness that the density is normalized $\int \mu_t(dx) \to 1$ as $t \to \infty$ [?, 8]. The expectation value can now be expressed in terms of periodic orbits as

$$< w >= \int w(x, t) \mu(dx) = \tag{53}$$

$$\lim_{t \to \infty} \sum_{p \in p.p.o.} T_p \sum_{n=1}^{\infty} w_p^n \frac{\delta(t - nT_p)}{|det(1 - M_p^n)|} \tag{54}$$

Comparing this expression with eq (5) immediately verifies eq (4).